

# Absolute continuity in periodic thin tubes and strongly coupled leaky wires

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**Abstract:** Using a perturbative argument, we show that in any finite region containing the lowest transverse eigenmode, the spectrum of a periodically curved smooth Dirichlet tube in two or three dimensions is absolutely continuous provided the tube is sufficiently thin. In a similar way we demonstrate absolute continuity at the bottom of the spectrum for generalized Schrödinger operators with a sufficiently strongly attractive  $\delta$  interaction supported by a periodic curve in  $\mathbb{R}^d$ ,  $d = 2, 3$ .

## 1 Introduction

In models of periodically structured quantum systems, absolute continuity of the spectrum is a crucial property. For usual Schrödinger operators and many other PDE's with periodic coefficients the problem is well understood

– see, e.g. [Ku, RS]. On the other hand, there are important classes of operators which still pose open questions. An example is represented by so-called *quantum waveguides*, i.e. systems the Hamiltonian of which is (a multiple of) the Laplacian in an infinitely long tube-shaped region, usually with Dirichlet boundary conditions.

In the two-dimensional setting, where the region in question is a periodically curved planar strip the absolute continuity has been demonstrated recently in [SV]. Unfortunately, the method used in this work does not seem to generalize to other dimensions including the physically interesting case of a periodic tube in  $\mathbb{R}^d$ ,  $d = 3$ . This is why we present in this letter a simpler result stating the absolute continuity at the bottom of the spectrum for tubes which are thin enough. With physical applications in mind we formulate it for  $d = 2, 3$ , but the argument can be used in any dimension.

We also address an analogous problem concerning Schrödinger operators in  $L^2(\mathbb{R}^d)$ ,  $d = 2, 3$ , with an attractive  $\delta$  interaction supported by a periodic curve; we prove absolute continuity, again at the bottom of the spectrum, for a sufficiently strong attraction. If  $d = 2$  the answer was known for a family of curves periodic in two independent directions [BŠŠ], however, for a single infinite curve results have been missing, to say nothing about the more singular case of dimension  $d = 3$ .

Our method is perturbative. We show that for a sufficiently thin tube or strongly attractive  $\delta$  interaction the Floquet eigenvalues do not differ much from the Floquet eigenvalues of a suitable comparison problem in one dimension which are known to be nonconstant as functions of quasimomentum. A similar argument was used recently for periodically perturbed magnetic channels [EJK], and in a different context to demonstrate existence of persistent currents in leaky quantum wire loops [EY2].

## 2 Thin curved tubes

Let  $\Gamma : \mathbb{R} \rightarrow \mathbb{R}^d$ ,  $d = 2, 3$ , be a  $C^4$  smooth curve without self-intersections which is periodic, i.e. there are  $L > 0$  and a nonzero vector  $b \in \mathbb{R}^d$  such that

$$\Gamma(s + L) = b + \Gamma(s), \quad \forall s \in \mathbb{R}.$$

With an abuse of notation we will use the same symbol  $\Gamma$  for the map and for the image  $\Gamma(\mathbb{R})$ . Let further  $\Omega := \{x \in \mathbb{R}^d : \text{dist}(x, \Gamma) < a\}$  for a fixed  $a > 0$ . If the latter is small enough, such a tube has no self-intersections and

we can parametrize it by natural curvilinear coordinates. More specifically, take a ball  $\mathcal{B}_a := \{u \in \mathbb{R}^{d-1}, |u| < a\}$  and consider the maps  $\phi : \mathbb{R} \times \mathcal{B}_a \rightarrow \mathbb{R}^d$  defined by

$$\begin{aligned}\phi_\Gamma(s, u) &= \Gamma(s) - un(s), \\ \phi_\Gamma(s, r, \vartheta) &= \Gamma(s) - r[n(s) \cos(\vartheta - \beta(s)) + b(s) \sin(\vartheta - \beta(s))]\end{aligned}$$

for  $d = 2, 3$ , respectively, where  $n(s), b(s)$  are the normal and binormal vector to  $\Gamma$  at the point  $s$  and the function  $\beta$  in the three-dimensional case (rotation with respect to the Frenet frame) will be specified later. We will always assume that  $a$  is so small that the above maps are diffeomorphisms.

The object of our interest is the Dirichlet Laplacian  $-\Delta_D^\Omega$  defined conventionally [RS, Sec. XIII.15] as the self-adjoint operator associated with the quadratic form  $\psi \mapsto \|\nabla \psi\|_{L^2(\Omega)}^2$  with the domain  $W_0^{2,1}(\Omega)$ .

The curvilinear coordinates allow us to “straighten” the tube, i.e. to pass to an operator on  $L^2(\mathcal{C}_a)$ , where  $\mathcal{C}_a := \mathbb{R} \times \mathcal{B}_a$  is a straight cylinder. The appropriate unitary operator  $U : L^2(\Omega) \rightarrow L^2(\mathcal{C}_a)$  is at that defined by  $U\psi := g^{1/4}\psi \circ \phi_\Gamma$ , where  $g^{1/2}$  is the corresponding Jacobian,  $g^{1/2} = 1 + u\gamma(s)$  for  $d = 2$  and  $g^{1/2} = 1 + r\gamma(s) \cos(\vartheta - \beta(s))$  for  $d = 3$ , with  $\gamma$  being the curvature of  $\Gamma$ . The price we pay for the simplification of the region is the more complicated form of the operator  $H_{a,\Gamma} := U(-\Delta_D^\Omega)U^{-1}$ ; a straightforward calculation [EŠ, DE] gives

$$H_{a,\Gamma} = -\partial_s h^{-2} \partial_s - \Delta_D^{\mathcal{B}_a} + V,$$

where  $h := g^{1/2}$  and  $V$  is the effective potential induced by the geometry,

$$V = -\frac{\gamma^2}{4h^2} + \frac{h_{ss}}{2h^3} - \frac{5h_s^2}{4h^4}.$$

For  $d = 3$  the transformation requires to choose  $\beta(s) = \int_{s_0}^s \tau(s) ds$ , where  $\tau$  is the torsion of  $\Gamma$ ; the latter appear also in the effective potential coming from the derivatives of  $h$  by Frenet-Serret formulae [DE].

As the first step we denote  $\mathcal{C}_a^L := [0, L) \times \mathcal{B}_a$  and perform the usual Floquet decomposition over the Brillouin zone  $\mathcal{B} := [-\pi/L, \pi/L)$ .

**Lemma 2.1** *There is a unitary  $\mathcal{U} : L^2(\mathcal{C}_a) \rightarrow \int_{\mathcal{B}}^\oplus L^2(\mathcal{C}_a^L) d\theta$  such that*

$$\mathcal{U} H_{a,\Gamma} \mathcal{U}^{-1} = \int_{\mathcal{B}}^\oplus H_{a,\Gamma}(\theta) d\theta,$$

where the fibre operator satisfies periodic b.c. in  $s$  acting as

$$H_{a,\Gamma}(\theta) = (-i\partial_s + \theta)h^{-2}(-i\partial_s + \theta) - \Delta_D^{\mathcal{B}_a} + V.$$

*Proof:* This is a classical result. We use here the modification, sometimes ascribed to Skriganov, of [RS, Thm XIII.88] with the Floquet-Bloch transform given for all  $(\theta, s, u) \in \mathcal{B} \times [0, L) \times \mathcal{B}_a$  by the formula

$$(\mathcal{U}f)(\theta, s, u) := \sum_{n \in \mathbb{Z}} \sqrt{\frac{L}{2\pi}} e^{-in\theta L - i\theta s} f(s + Ln, u).$$

An exhaustive discussion for  $d = 2$  can be found in [Yo], the argument for  $d = 3$  is analogous. ■

We need also the character of  $\theta$ -dependence of the fibre operators.

**Lemma 2.2**  *$\{H_{a,\Gamma}(\theta) : \theta \in \mathcal{B}\}$  is a type A analytic family.*

*Proof:* Splitting  $H_{a,\Gamma}(\theta)$  in two pieces  $H_{a,\Gamma}(0) = -\partial_s h^{-2} \partial_s - \Delta_D^{\mathcal{B}_a} + V$  and  $H_{a,\Gamma}(\theta) - H_{a,\Gamma}(0) = -i\theta(\partial_s h^{-2} + h^{-2} \partial_s) + \theta^2 h^{-2}$ , one sees that the first piece is self-adjoint on  $\{f \in W^{2,2}(\mathcal{C}_a^L), f(L, u) = f(0, u), \partial_s f(L, u) = \partial_s f(0, u)\}$  and that the second one is entire analytic and relatively bounded perturbation of  $H_{a,\Gamma}(0)$  with zero relative bound. ■

Our main tool is the perturbation theory w.r.t.  $a$ . Since the argument follows closely [DE] we will just sketch it. First we use transverse scaling to pass to a unitarily equivalent operator on  $L^2(\mathcal{C}_a^L)$  given by

$$\tilde{H}_a(\theta) := (-i\partial_s + \theta)h_a^{-2}(-i\partial_s + \theta) - a^{-2}\Delta_D^{\mathcal{B}_1} + V_a,$$

where  $h_a(s, u) := h(s, au)$  and  $h_a(s, r, \vartheta) := h(s, ar, \vartheta)$  for  $d = 2, 3$ , respectively, and similarly for  $V_a$ . Let  $\chi_j$  and  $\kappa_j^2$  be the eigenfunctions and eigenvalues of  $-\Delta_D^{\mathcal{B}_1}$ . Using this transverse basis we pass to the matrix representation,

$$\tilde{H}_{a,jk}(\theta) = a^{-2}\kappa_j^2\delta_{jk} + T_{jk}, \quad T_{jk} := (-i\partial_s + \theta)(h_a^{-2})_{jk}(-i\partial_s + \theta) + V_{a,jk},$$

where  $f_{jk} := \int_{-1}^1 f(\cdot, au) \chi_j(u) \chi_k(u) du$  if  $d = 2$  and similarly for  $d = 3$ . We need a reference operator which will be chosen in the form

$$\tilde{H}_a^0(\theta) := I \otimes (-a^{-2}\Delta_D^{\mathcal{B}_1}) + S(\theta) \otimes I, \quad S(\theta) := (-i\partial_s + \theta)^2 - \frac{1}{4}\gamma^2$$

with periodic b.c. in the variable  $s$ . The spectrum of  $S(\theta)$  is purely discrete; we denote its eigenvalues arranged in the ascending order as  $\lambda_n(\theta)$  with the

index  $n \in \mathbb{N}$ . Recall that the spectrum of  $S(\theta)$  is simple with a possible exception of the endpoints of the Brillouin zone and  $\theta = 0$ . Let  $K$  be a compact subset of  $\mathcal{B}$  which does not contain the points 0 and  $\pm\pi/L$ . The eigenvalues  $\epsilon_{jn}^0(a, \theta) := a^{-2}\kappa_j^2 + \lambda_n(\theta)$  of  $\tilde{H}_a^0(\theta)$  are isolated and of finite multiplicity; with the above choice of  $K$  they even become simple on  $K$  for any fixed  $n$  and  $j = 1$  if  $a$  is small enough. They depend on  $a$ , of course, but one can perform the perturbation expansion with respect to  $W(\theta) := H_a(\theta) - H_a^0(\theta)$  around such running values. Mimicking the argument of [DE] we come to the following conclusion.

**Lemma 2.3** *Let  $n_0 \in \mathbb{N}$  and  $E_{n_0} := ((2n_0 - 1)\pi/L)^2$ . There exists a positive  $a_{K, n_0}$  such that for all  $a \in (0, a_{K, n_0})$  and any  $\theta \in K$ , the spectrum of  $\tilde{H}_a(\theta)$  below  $E_{n_0}$  consists of exactly  $n_0$  simple eigenvalues  $\{\epsilon_{1,n}(a, \theta)\}_{1 \leq n \leq n_0}$ . Moreover, the expansion*

$$\epsilon_{1,n}(a, \theta) = a^{-2}\kappa_1^2 + \lambda_n(\theta) + \mathcal{O}(a)$$

*holds for each  $n = 1, \dots, n_0$  uniformly in  $K$ .*

Armed with these prerequisites, we can now formulate and prove the main result of this section.

**Theorem 2.4** *To any  $E > 0$  there is  $a_E > 0$  such that the spectrum of  $-\Delta_D^\Omega$  is absolutely continuous in the interval  $[0, a^{-2}\kappa_1^2 + E]$  for all  $a < a_E$ .*

*Proof:* By Lemma 2.1 we have to check that the eigenvalues of  $\tilde{H}_a(\theta)$  are nowhere constant as functions of the quasimomentum  $\theta$ . Since they are real-analytic by Lemma 2.2, we have only to verify that each eigenvalue branch acquires at least two different values in the set  $K$ . Without loss of generality we can put  $E = E_{n_0}$  for some  $n_0 \in \mathbb{N}$ . Then there is just  $n_0$  eigenvalues in  $[0, a^{-2}\kappa_1^2 + E]$ , and consequently, there exists a  $c_E > 0$  such that  $|\epsilon_{1,n}(a, \theta) - a^{-2}\kappa_1^2 + \lambda_n(\theta)| \leq c_E a$  holds for each of them according to Lemma 2.3. Since the functions  $\lambda_n(\cdot)$  are non-constant by [RS, Sec. XIII.16], the conclusion follows. ■

**Remark 2.5** A similar perturbative argument shows that the spectrum is absolutely continuous at its bottom if  $a$  is fixed and  $\|\gamma\|_\infty$  is sufficiently small.

### 3 Leaky quantum wires

Now we are going to consider the analogous problem for another class of operators. Let  $\Gamma$  be the same  $C^4$ -smooth periodic curve in  $\mathbb{R}^d$  and consider the generalized Schrödinger operator given by the formal expression

$$H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x - \Gamma).$$

Its meaning is different for different dimensions. If  $d = 2$  we regard it as the unique self-adjoint operator associated with the quadratic form

$$q_{\alpha,\Gamma}[\psi] = \|\nabla\psi\|^2 - \alpha \int_{\mathbb{R}} |\psi(\Gamma(s))|^2 ds, \quad \psi \in W^{2,1}(\mathbb{R}^2),$$

which is closed and below bounded by [BEKŠ]; we suppose that the singular interaction is attractive,  $\alpha > 0$ . The situation is more complicated in the three-dimensional case when  $\text{codim } \Gamma = 2$ . Following the construction given in [EK1] – see also [Po] for a more general background – one starts from the family of curves  $\phi_\Gamma(\cdot, \rho, \vartheta_0)$ , obtained by translating  $\Gamma$  by  $\rho(\cos(\vartheta_0), \sin(\vartheta_0))$ , which are used to determine the generalized boundary values of a function  $f\psi \in W_{\text{loc}}^{2,2}(\mathbb{R}^3 \setminus \Gamma) \cap L^2(\mathbb{R}^3)$  as the following limits

$$\begin{aligned} L_0(\psi)(s) &:= -\lim_{\rho \rightarrow 0} \frac{1}{\ln \rho} \psi(\phi_\Gamma(s, \rho, \vartheta_0)), \\ L_1(\psi)(s) &:= \lim_{\rho \rightarrow 0} [\psi(\phi_\Gamma(s, \rho, \vartheta_0)) + L_0(\psi)(s) \ln \rho]. \end{aligned}$$

We call  $\Upsilon_\Gamma$  the family of those  $\psi$  for which these limits exist a.e. in  $\mathbb{R}$ , are independent of  $\vartheta_0$ , and define a pair functions belonging to  $L_{\text{loc}}^2(\mathbb{R})$ . The sought operator is then defined as

$$\begin{aligned} H_{\alpha,\Gamma}\psi(x) &:= -\Delta\psi(x), \quad x \in \mathbb{R}^3 \setminus \Gamma, \\ D(H_{\alpha,\Gamma}) &:= \{f \in \Upsilon_\Gamma : 2\pi\alpha L_0(\psi)(s) = L_1(\psi)(s)\}. \end{aligned}$$

Being interested in strong coupling we suppose hereafter that the coupling parameter  $\alpha$  is negative though  $H_{\alpha,\Gamma}$  is well defined for all  $\alpha \in \mathbb{R}$ ; recall that the two-dimensional  $\delta$  interaction is always attractive [AGHH].

In an important particular case when  $\Gamma$  is a straight line one uses separation of variables to show that the spectrum is absolutely continuous and

covers the interval  $[\zeta(\alpha), \infty)$ , with the threshold given by the corresponding  $(d-1)$ -dimensional  $\delta$  interaction eigenvalue,

$$\zeta(\alpha) := \begin{cases} -\frac{1}{4}\alpha^2 & \dots & d = 2 \\ -4e^{2(-2\pi\alpha + \psi(1))} & \dots & d = 3 \end{cases}$$

where  $-\psi(1) \approx 0.577$  is the Euler number. This also illustrates that the strong coupling means  $(-1)^d \alpha \rightarrow \infty$  for  $d = 2, 3$ . Due to the injectivity and periodicity assumptions the curve decomposes into a disjoint union of translates of the period cell  $\Gamma_{\mathcal{P}} := \Gamma \upharpoonright [0, L)$ . Since  $H_{\alpha, \Gamma}$  now acts in the whole Euclidean space, we need also a decomposition of the space  $\mathbb{R}^d$  with the period cell

$$\mathcal{P} := \{ \mathcal{L} + tb : t \in [0, 1) \} ,$$

where  $\mathcal{L} \subset \mathbb{R}^d$  is a affine space which is not colinear with  $b$ . We denote by  $b_{\perp}$  the component of  $b$  in the direction orthogonal to  $\mathcal{L}$ ; it follows that  $b_{\perp} \neq 0$ . It is important that the two decompositions are chosen in a consistent way,  $\Gamma_{\mathcal{P}} = \mathcal{P} \cap \Gamma$ . We will assume in addition that

(c) the restriction of  $\Gamma_{\mathcal{P}}$  to the interior of  $\mathcal{P}$  is connected.

It should be noted that the choice of a slab for  $\mathcal{P}$  is made rather for convenience – see Remarks 3.5 below. We start again from the Floquet decomposition with respect to the Brillouin zone  $\mathcal{B} := [-\pi|b_{\perp}|^{-1}, \pi|b_{\perp}|^{-1})$ .

**Lemma 3.1** *There is a unitary  $\mathcal{U} : L^2(\mathbb{R}^d) \rightarrow \int_{\mathcal{B}}^{\oplus} L^2(\mathcal{P}) d\theta$  such that*

$$\mathcal{U} H_{\alpha, \Gamma} \mathcal{U}^{-1} = \int_{\mathcal{B}}^{\oplus} H_{\alpha, \Gamma}(\theta) d\theta ,$$

where the fibre operator satisfies periodic b.c. in the direction of  $b$  acting as

$$H_{a, \Gamma}(\theta) = (-i\nabla + \theta)^2 - \alpha\delta(x - \Gamma) ,$$

and the interaction term in  $L^2(\mathcal{P})$  is interpreted in the above described sense, the quadratic form if  $d = 2$  and boundary conditions if  $d = 3$ .

*Proof:* See [EY1] for  $d = 2$  and [EK2] for  $d = 3$ . ■

It is easy to see that in distinction to the previous case the essential spectrum is non-empty and equals  $\sigma_{\text{ess}}(H_{a, \Gamma}(\theta)) = [\theta^2, \infty)$ ; we will be interested in the eigenvalues below its threshold. They are again real-analytic functions:

**Lemma 3.2**  $\{H_{\alpha,\Gamma}(\theta) : \theta \in \mathcal{B}\}$  is a type A analytic family.

*Proof:* Similar to that of Lemma 2.2. ■

The role of the small tube width from the previous section is played here by strong coupling. While the wave function may be nonzero at large distances from  $\Gamma$ , it is localized in its vicinity as  $(-1)^d \alpha \rightarrow \infty$ . Then one can choose a tubular neighbourhood  $\Omega$  of  $\Gamma$  and estimate the operator in question from both sides by imposing the Dirichlet and Neumann condition at  $\partial\Omega$ . The exterior part does not contribute to the negative spectrum, while the part in  $\Omega$  can be treated as in the previous section, with the additional  $\delta$  interaction on the tube axis and different boundary conditions for the lower bound. The argument is thus more complicated, however, it was done in [EY1] and [EK2] with the following result.

**Lemma 3.3** *The number of isolated eigenvalues of  $H_{\alpha,\Gamma}(\theta)$  exceeds any fixed  $n \in \mathbb{N}$  as  $(-1)^d \alpha \rightarrow \infty$ . The  $n$ -th eigenvalue behaves asymptotically as*

$$\epsilon_n(\alpha, \theta) := \zeta(\alpha) + \lambda_n(\theta) + \left\{ \begin{array}{l} \mathcal{O}(\alpha^{-1} \ln \alpha) \\ \mathcal{O}(e^{\pi\alpha}) \end{array} \right\}$$

*in the strong coupling limit for  $d = 2, 3$ , respectively, uniformly in  $\theta$ , i.e. the error terms is for a fixed  $n$  bounded in  $\mathcal{B}$ . Here  $\lambda_n(\theta)$  means again the  $n$ -th eigenvalue of the operator  $S(\theta)$  defined in the previous section.*

The main result of this section then reads as follows.

**Theorem 3.4** *Under the stated assumptions, to any  $E > 0$  there exists an  $\alpha_E > 0$  such that the spectrum of the operator  $H_{\alpha,\Gamma}$  is absolutely continuous in  $(-\infty, \zeta(\alpha) + E]$  as long as  $(-1)^d \alpha > \alpha_E$ .*

*Proof:* The argument is analogous to the one used for Theorem 2.4. ■

**Remarks 3.5** (i) It is not always possible to choose  $\mathcal{P}$  is the Cartesian-product form, as we did above, which would satisfy the assumption (c). Counterexamples with a sufficiently entangled periodic  $\Gamma$  are easily found. However, if we choose instead another period cell  $\mathcal{P}$  with a smooth boundary, which is not a planar slab and for which the property (c) is valid, the argument modifies easily and the claim of Theorem 3.4 remains true.



(ii) In the case  $d = 2$  such a “puzzle-like” decomposition can be always found. To see that, fold the plane into a cylinder of radius  $|b|/2\pi$  so  $\Gamma$  becomes a loop which encircles the cylindrical surface, dividing into two parts  $\mathcal{C}^\pm$  which are disjoint apart of the common boundary; each of them is connected because  $\Gamma$  has by assumption no self-intersections. Choosing a point at  $\Gamma$ , one can thus find two smooth semi-infinite curves in  $\mathcal{C}^\pm$ , even straight from some point on, which go the two cylinder “infinities” without crossing  $\Gamma$ .

(iii) On the other hand, an analogous decomposition into translates of a suitable  $\mathcal{P}$  satisfying the hypothesis (c) may not exist if  $d = 3$ . It depends on the topology of  $\Gamma$ , a simple counterexample is given by a “crotchet-shaped” curve which enters  $\mathcal{P}$  on its “left side” twice and leaves it once, and vice versa on the right, without being topologically equivalent to a line. We conjecture that the claim of Theorem 3.4 remains valid in such situations too, however, a different method is required to demonstrate it.

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